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16. Abstract

The problem of optimal speed of response will not seek a control function, but the optimal selection of velocity from the set V. If some trajectory is realized by both systems of equations contained in this article, with an identical increase in the argument and is optimal for the second system, it is also optimal for the initial system.

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1. Statement of the Problem of Optimal Speed of Response /57

Let the point (x_1, x_2) of plane R^2 correspond to the state of some system, and the motion of the system be described by the equations

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, u), \\ \dot{x}_2 &= f_2(x_1, x_2, u), \end{aligned} \tag{1.1}$$

where (•) designates the differentiation by argument t, u--controls acquiring the value from some subset U of Euclidian space. The functions f_1 and f_2 are assumed continuous together with the first partial derivatives with respect to the variables x_1 and x_2. We must select such piecewise continuous vector-function u(t) whereby the system transfers from the initial state to the origin of the coordinates with the least increment in the argument. Since in practice, the argument most often is time, then the formulated problem has received the name of a problem of optimal speed of response.

Equations (1.1) for each point (x_1, x_2) assign the mapping of the set U on the plane. Let us designate by V(x_1, x_2) the image of the set U in such mapping. The problem of optimal speed of response can be stated differently. Namely, to put in correspondence to each point (x_1, x_2) a set V(x_1, x_2). In this case, we will not be seeking a control function, but the optimal selection of velocity from the set V. Henceforth, both methods of stating the problem will be used.

Let us examine, together with the initial system with equations (1.1), another system with equations defining at each point (x_1, x_2) a set $V_1(x_1, x_2)$. Let us plot the limitation in phase coordinates: $(x_1, x_2) \in X$, where the set X is previously selected. Now let for every $(x_1, x_2) \in X$ /58

$$V(x_1, x_2) \subset V_1(x_1, x_2). \tag{1.2}$$

Then, if some trajectory is realized by both systems with an identical increase in the argument and is optimal for the second system, it is also optimal for the initial system.

Actually, condition (1.2) signifies that the second system in any direction can move with less velocity than the initial, whence follows the assertion.

2. Definition of an Auxiliary System

Let us consider the problem of speed of response for a system whose set V for every point $(x_1, x_2) \in X$ is a band contained between two parallel straight lines, and point (0, 0) is an internal point V. Therefore, for this system at each point of the set X there is a direction in which it can move without expenditure of the argument. In other words, after defining the system, we defined the field of directions possessing this property.

Now let m be an integral curve of this field of directions and let m divide the set X into two subsets, so that it is impossible to go from one to another continuously without intersecting m. Then we can predict a priori on which of the straight lines defining the set V lies the optimal value of velocity for points on curve m.

Actually, since (0, 0) is the internal point of the band, then by selecting the value of velocity on one of the straight lines, we enter that part of set X to which belongs the ultimate point. The selection of velocity on another straight line corresponds to a departure from curve m into an area to which the ultimate point does not belong. The selection of velocity in the second instance can not be optimal, since for us to enter into the finite point, we still have to intersect curve m, expending on this action some increase in the argument, whereas motion along curve m occurs without loss of argument.

So, this system is tantamount to a system whose set V is a straight line p: $P_1(x_1, x_2)x_1 + P_2(x_1, x_2)x_2 = 1$. This fact will be used later to fulfill condition (1.2).

Let functions P_1 and P_2 be piecewise continuous together with their first partial derivatives and ΔP_1, ΔP_2--their increments in transition through the curve of discontinuity. Then, if everywhere beside the curves of discontinuity is fulfilled the relation /59

$$\frac{\partial P_2}{\partial x_1}=\frac{\partial P_1}{\partial x_2}, \qquad (2.1)$$

and on the curves of discontinuity is a relation

$$\Delta P_1\,dx_1+\Delta P_2\,dx_2=0, \qquad (2.2)$$

where dx_1, dx_2 define the direction tangent to the curve of discontinuity, the problem of optimization for this system is degenerated. Namely any trajectory which can be realized is optimal.

Let us now return to the initial problem. Let each point of X be placed in correspondence with a set V and solve the problem of speed of response. Let us assume that we were able to construct a degenerate system (i.e., a system for which are fulfilled conditions (2.1) and (2.2) and the condition on curve m) and such that the straight line p is a line of support of the convex envelope V. This system will be called an auxiliary. It is now not difficult to make an optimal selection of velocity: the optimal values of velocity belong to the intersection of straight line p and convex envelope V.

3. Example for Construction of Auxiliary System

Let us consider the problem of speed of response for a system with controls

$$\dot{x}_1 = \frac{u}{\varphi_1(x_1, x_2)}, \qquad \dot{x}_2 = \frac{1-u}{\varphi_2(x_1, x_2)}, \tag{3.1}$$

where $u \in [0, 1]$ and $\phi_1, \phi_2 > 0$. The set V for this system is a fragment whose ends lie on half-lines $\dot{x}_2 = 0$, $\dot{x}_1 > 0$ and $\dot{x}_1 = 0$, $\dot{x}_2 > 0$. This system can be translated to the origin of the coordinates only from the third quadrant of the plane; consequently, the auxiliary system must be constructed only in that quadrant. Let us note that the construction of an auxiliary system is equivalent (in this case) to the construction of a synthesis of optimal control, i.e., to the definition of control as a function of phase coordinates.

Let us construct the auxiliary system in the small neighborhood of the origin of the coordinates, more precisely on the intersection of the δ neighborhood and the third quadrant of the plane; we will then expand the structure to the remaining area.

Let us consider the function $\omega(x_1, x_2) = \frac{\partial \phi_2}{\partial x_1} - \frac{\partial \phi_1}{\partial x_2}$. By de- /60 finition, this function is continuous. Let (0, 0) < 0. Let us then assume that

$$\begin{aligned} P_1 &= \varphi_1, \\ P_2 &= \varphi_2 - \int_0^{x_1} \omega(y, x_2)\,dy. \end{aligned} \qquad (3.2)$$

We will show that the system defined by formulas (3.2) is auxiliary in some small neighborhood of the origin of the coordinates.

Actually, functions P_1 and P_2 satisfy conditions (2.1) and (2.2), and due to the continuity of function ω there exists a neighborhood in which $P_2 < \phi_2$; consequently, condition (1.2) is also fulfilled. Curves m are defined by the equation

$$P_1\,dx_1 + P_2\,dx_2 = 0, \qquad (3.3)$$

whence is verified the condition imposed on m and line p.

After constructing the auxiliary system, it is easy to define the control as a function of the variables x_1 and x_2. Namely $u = 1$, if $x_2 < 0$, and $u = 0$ if $x_2 = 0$.

Let $\omega(0, 0) > 0$. Then let us assume that

$$\begin{aligned} P_1 &= \varphi_1 + \int_0^{x_2} \omega(x_1, y)\,dy, \\ P_2 &= \varphi_2. \end{aligned} \qquad (3.4)$$

By applying reasoning analogous to the preceeding, it is easy to show that functions P_1 and P_2 define the auxiliary system. The optimal control is defined as follows: $u = 0$ if $x_2 < 0$, $u = 1$ if $x_2 = 0$.

Now let $\omega(0, 0) = 0$. Construction of the auxiliary system depends on which set of points defines in the neighborhood of the origin of the coordinates the equation $\omega(x_1, x_2) = 0$. The continuity of function ω yields, in this case, insufficient information on this set. Thus, we will impose additional restrictions on function ω.

Let curves l_i ($i = 1, \ldots, k$), intersecting only at the origin of the coordinates, cut across neighborhood δ into sectors r_i. Sector r_i is bounded by curves l_i and l_{i+1}, sector r_1 is bounded by straight line $x_1 = 0$ and curve l_1; and sector r_{k+1}--by curve l_k and straight line $x_2 = 0$. Curves l_i can be given with the aid of equations $x_1 = \psi_i(x_2)$ or $x_2 = \chi_i(x_1)$, where ψ_i and χ_i are continuous and monotonous functions, defined for sufficiently small values of the argument; and $\psi_k > \psi_{k-1} > \ldots > \psi_1$, and $\chi_k < \chi_{k-1} < \ldots < \chi_1$.

We will consider that we can always construct a breakdown of neighborhood δ, which has the aforementioned properties so that in each sector r_i, function ω is either non-positive or non-negative, and in transition through curve l_i the function would change sign. /61

Let us construct an auxiliary system based on this supposition. Let us introduce into our discussion a set of a finite number of curves n_i passing through the origin of the coordinates, after defining them as follows. Curve n_i coincides with curve l_i if in transition from sector r_i to sector r_{i+1} function ω changes sign plus to minus. If the sign changes in the opposite direction, curve n_i is defined by the equation

$$\left(\int_{\psi_{i+1}(x_2)}^{x_1} \omega(y, x_2)\, dy\right) dx_2 = \left(\int_{\chi_i(x_1)}^{x_2} \omega(x_1, y)\, dy\right) dx_1. \qquad (3.5)$$

Curves n_i define a new subdivision of the neighborhood into sectors s_i. Sector s_1 is bounded by straight line $x_1 = 0$ and curve n_1; sectors s_i $(i \neq 1)$--by curves n_i and n_{i+1}; sector s_{k+1} by curve n_k and straight line $x_2 = 0$.

Now let $(x_1, x_2) \in s_i (i \neq 1)$. If curve n_i coincides with curve l_i, then we assume that

$$\boxed{\begin{aligned} &P_1=\varphi_1, \\ &P_2=\varphi_2-\int_{\psi_i(x_1)}^{x_2} \omega(y, x_2)\, dy. \end{aligned}} \tag{3.6}$$

If curve n_i, however, does not coincide with curve l_i, then accordingly curve n_{i+1} coincides with curve l_{i+1}. Then we will assume that

$$\boxed{\begin{aligned} &P_1=\varphi_1+\int_{\chi_{i+1}(x_1)}^{x_2} \omega(x_1, y)\, dy, \\ &P_2=\varphi_2. \end{aligned}} \tag{3.7}$$

Let $(x_1, x_2) \in s_1$. If n_1 coincides with l_1, we assume that

$$\begin{aligned} &P_1=\varphi_1+\int_{\chi_1(x_1)}^{x_1} \omega(x_1, y)\, dy. \\ &P_2=\varphi_2, \end{aligned} \tag{3.8}$$

and if the condition is not fulfilled, then

$$\boxed{\begin{aligned} &P_1=\varphi_1, \\ &P_2=\varphi_2-\int_0^{x_1} \omega(y, x_2)\, dy. \end{aligned}} \tag{3.9}$$

Let $(x_1, x_2) \in s_{k+1}$. If n_k coincides with l_k, then

$$\boxed{\begin{aligned} &P_1=\varphi_1+\int_{\chi_k(x_1)}^{x_2} \omega(x_1, y)\, dy \\ &P_2=\varphi_2, \end{aligned}} \tag{3.10}$$

in the opposite situation we find that

$$P_1 = \varphi_1, \quad P_2 = \varphi_2 - \int_0^{x_1} \omega(y, x_2)\, dy. \tag{3.11}$$

After applying notions analogous to the case $\omega(0, 0) < 0$, it is not difficult to convince ourselves that formulas (3.6)-(3.11) define the auxiliary system.

Therefore, after imposing the restriction on function ω, we were able to solve the problem in the neighborhood of the origin of the coordinates. Let us now try to expand the found construction to a wider area. Let us note that if the system is defined in the form

$$P_1 = \varphi_1 + \int_{g_1(x_1)}^{x_2} \omega(x_1, y)\, dy, \quad P_2 = \varphi_2. \tag{3.12}$$

or as

$$P_1 = \varphi_1, \quad P_2 = \varphi_2 - \int_{g_2(x_2)}^{x_1} \omega(y, x_2)\, dy. \tag{3.13}$$

where g_1 and g_2 are differentiable functions, then condition (2.1) is fulfilled. Accordingly, it remains to select such functions g_1 and g_2 and such curves of discontinuity of functions P_1 and P_2 which would enable conditions (1.2) and (2.2) to be fulfilled.

Reversing curves will be called those curves given by the equations $x_1 = g_2(x_2)$ and $x_2 = g_1(x_1)$. In the reversing curves the relations are fulfilled as follows

$$P_1 = \varphi_1, \quad P_2 = \varphi_2. \tag{3.14}$$

with whose aid we can construct these curves. The construction begins the moment when condition (1.2) is threatened by nonfulfillment.

The curves of discontinuity are constructed with the aid of relation (2.2) and serve to define functions P_1 and P_2 unequivocally.

4. More General Case of Problem of Speed of Response

Let us consider the problem of speed of response for a system which has a set of V that is a convex polygon containing a zero point. Let d be the number of polygon peaks and let this number not change in the neighborhood of the origin of the coordinates Considering the control point a fixed peak of the polygon, let us construct d curves of q_j which cut the neighborhood into d sectors. In each sector we will construct an auxiliary system, taking the corresponding side of the polygon as the set of V. This problem can be reduced to problem #3 with the aid of the appropriate transformation of coordinates. It turns out that the auxiliary systems constructed independently in each sector define the total auxiliary system for the problem of this section. /63

In reality, due to the convexity of the polygon, condition (1.2) is fulfilled. Curves q_j are curves of discontinuity, but they satisfy condition (2.2). Within the sectors, conditions (2.1) and (2.2) are fulfilled from the construction.

In expanding the construction to a wider area, we must keep to the same rules as in section 3.

Let us note in concluding that there is no theoretical difficulty in constructing an auxiliary system, if the functions defining the set V are only piecewise smooth with respect to the phase coordinates.